PRINCIPLES AND METHODS OF THE CALCULATION OF TRANSFER CHARACTERISTICS OF DISK PIEZOELECTRIC TRANSFORMERS

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ABSTRACT

Context. Thanks to its unique properties piezoceramics has applications in various fields of engineering and technology. Disk piezoelectric devices are widely used in the elements of information systems. Research has shown that piezoelectric transformers can compete with traditional electromagnetic transformers on both efficiency and power density. The final goal of mathematical modeling of the vibrating piezoelectric elements physical condition is a qualitative and quantitative description of characteristics and parameters of existing electrical and elastic fields.

Objective. The purpose of this paper is to set out the principles of mathematical models construction that are sufficiently adequate to real devices and occurring physical processes using the simplest example of axially symmetric radial oscillations of the piezoelectric disk.

Method. Mathematical models of piezoelectric transformers working with axially symmetric radial oscillations of piezoceramic disks are constructed with a minimal number of assumptions simplifying the real situation. This allows us to state that the proposed construction scheme delivers mathematical models that are sufficiently adequate to the real objects and physical processes that exist in them.

Results. Main results of this work can be formulated as follows: mathematical model of piezoelectric transformer with ring electrode in the primary electrical circuit is constructed; high sensitivity of frequency characteristic of piezoelectric transformer to the values of the output impedance of the electrical signal source in the primary electrical circuit is demonstrated.

Conclusions. As a result of research of real device’s mathematical model a set of geometrical, physical and mechanical and electrical parameters of a real object can be determined which provides realization of technical parameters of piezoelectric functional element specified in technical specifications. The cost of the saved resources is the commercial price of the mathematical model. Prospects for further research can be to build a mathematical model of a piezoelectric transformer with sector electrodes.

KEYWORDS: piezoelectric transformer, axially symmetric oscillations, physical processes, mathematical model.

NOMENCLATURE

\( U_1 \) is an amplitude value of electric potential difference;

\( i = \sqrt{-1} \) is an imaginary unit;

\( \omega \) is an angular frequency;

\( t \) is a time;

\( x_k \) are coordinates of the point, in which it is determined the displacement of the piezoelectric material particles from the equilibrium position;

\( H \) is a set of geometrical and physical and mechanical properties of the piezoelectric transformer;

\( Z_1 \) is an electrical impedance of the input electrode 1;

\( I_1 \) is an amplitude value of the electric current in the conductor, which connects an input electrode 1 with a source of the electrical signals;

\( 2d_1 \) is a width of the input ring electrode;

\( \vec{B}, \vec{D} \) are vectors of the magnetic and electric induction of the electromagnetic field components;

\( u_r, u_z \) are amplitude values of the radial and axial components of the material particles displacement vector of dynamically deformable piezoelectric disk.

INTRODUCTION

Thanks to its unique properties piezoceramics has applications in various fields of engineering and
technology. The relevance of the use of various functional elements of piezoelectronics in radio devices, information and power systems is explained by their high reliability and small dimensions, which solves the problem of miniaturization of such systems. Piezoelectric disks with surfaces partially covered electrodes are often used to create various functional piezoelectronic devices. Disk piezoelectric devices are widely used in the elements of information systems. In disk piezoelectric elements with surfaces partially covered by electrodes we can simultaneously excite oscillations of compression-tension and transverse bending vibrations. Manipulating the geometric parameters of electrodes and their location relative to each other, you can have a significant effect on the energy of oscillatory motion particular type of material particles of piezoelectric disk volume. It should be especially noted that this piezoelectric element has compatibility with microsystem technology, so it can be made as microelectromechanical structures (MEMS) [1]. One of the main elements of functional piezoelectronics is piezoelectric transformer (PT). Research has shown that PTs can compete with traditional electromagnetic transformers on both efficiency and power density [2–4]. PTs are therefore an interesting field of research [5]. The favorable attributes of the PT are low weight and size and potentially low cost. One additional important characteristic is the high voltage isolation of the ceramic materials used to build PTs [6]. In addition, a piezoelectric transformer is more suitable for mass production than traditional, coil-based transformers [7].

1 PROBLEM STATEMENT

The operation principle of piezoelectric transformers is generally known [8].

When applying an electrical potential difference \( U_1 e^{it} \) to pair of electrodes that are partially cover the front and bottom surfaces of the piezoelectric plate, harmonic oscillations of material particles are excited in a volume of the plate, which, in general, can be described by the displacement vector of material particles \( u(x_k) e^{it} \). Fluctuations of material particles are accompanied by dynamic deformations \( \varepsilon_{mn}(x_k) e^{it} \) of infinitely small elements of a piezoelectric volume. Due to the direct piezoelectric effect the harmonically varying in time according to \( e^{it} \) polarization charges with a surface density \( q_m(x_k) e^{it} \) arise in a deformable piezoelectric. Some of these charges are collected by the second pair of electrodes, which like the first pair, partially covers the surface of the piezoelectric plate. The polarization charge on the second pair of electrodes causes an electric current \( i(t) = I e^{it} \) in the conductor, which connects one of the electrodes of the second pair to the load impedance \( Z_n \). The voltage drop \( U_2 e^{it} = Z_n I e^{it} \) is an output signal of the piezoelectric transformer. Obviously, the transformation ratio \( K(\omega, \Pi) \) is equal to the ratio of the output signal to the input one, i.e.

\[
K(\omega, \Pi) = \frac{U_2}{U_1} = \frac{Z_n I}{U_1},
\]

and is a mathematical model of a piezoelectric transformer [9].

The practical value of the analytical structure \( K(\omega, \Pi) \) that adequately describes the physical processes in the real object is evident.

2 REVIEW OF THE LITERATURE

Many publications have been devoted to the construction and research of mathematical models of piezoelectric transformers. Starting with the monograph [8], the basics of the calculation of piezoelectric transformers’ transfer characteristics were considered, for example, in [10–13].

However, in many papers only processes occurring in a piezoelectric disk with a surface, fully covered by electrodes, are described. There are also a number of works of a disparate character devoted to the solution of the problem of electromechanical oscillations of piezoelectric elements with separated electrodes (transformer type). The constructions of piezoelectric transformer of a planar transverse-longitudinal and rod type are considered in [10] and [11], respectively. In [12] the analysis of the dependence of transformation coefficient of disk piezoelectric transformer on the location of secondary electrode, on the width of secondary electrode, and on the value of electrical load applied to secondary electrode was made. In [13] the radial axisymmetric oscillations of thin piezoceramic disk with a surface, partially covered by electrodes, are considered.

In many papers [14–19] the described methods of piezoelectric transformers models constructing are mostly based on the use of equivalent electrical circuits and it does not allow analyzing of stress-strain state of solids with the piezoelectric effects.

Based on the above, it can be argued that currently there are no reliable and valid methods of constructing of mathematical models of piezoelectric transformers, which could be used as a theoretical basis for characteristics and parameters calculating of this class of functional elements of modern piezoelectronics.

The purpose of this paper is to set out the principles of mathematical models construction that are sufficiently adequate to real devices and occurring physical processes using the simplest example of axially symmetric radial oscillations of the piezoelectric disk.

3 MATERIALS AND METHODS

Let us consider the disk with the radius \( R \) and the thickness \( \alpha \) (Fig. 1) made of piezoelectric ceramics PZT with thickness polarization during its manufacture i.e. along the coordinate axis \( z \) of the cylindrical coordinate
Electric polarization direction defines the properties and the matrices construction of piezoceramic disk’s material constants.

Figure 1 – Diagram of the piezoelectric disk transformer that operates on radial vibrations

The matrix of elastic moduli of piezoceramic disk polarized across the thickness looks like

\[
\begin{bmatrix}
  e^{E}_{11} & e^{E}_{12} & e^{E}_{13} & 0 & 0 & 0 \\
  e^{E}_{12} & e^{E}_{22} & e^{E}_{23} & 0 & 0 & 0 \\
  e^{E}_{13} & e^{E}_{23} & e^{E}_{33} & 0 & 0 & 0 \\
  0 & 0 & 0 & e^{E}_{44} & 0 & 0 \\
  0 & 0 & 0 & 0 & e^{E}_{55} & 0 \\
  0 & 0 & 0 & 0 & 0 & e^{E}_{66}
\end{bmatrix}
\]

where \( \lambda, \beta = 1; \ldots; 6 \) are Voigt indices; \( e^{E}_{11} = e^{E}_{22} \neq e^{E}_{33} \);

\( e^{E}_{12} = e^{E}_{13} = e^{E}_{23} ; e^{E}_{44} = e^{E}_{55} ; e^{E}_{66} = \left( e^{E}_{11} - e^{E}_{12} \right) / 2 \).

The matrix of piezomoduli \( e^{k\beta}_{k\beta} \) can be written as follows [10]

\[
\begin{bmatrix}
  0 & 0 & 0 & 0 & e^{k\beta}_{15} & 0 \\
  0 & 0 & 0 & e^{k\beta}_{24} & 0 & 0 \\
  e^{k\beta}_{31} & e^{k\beta}_{32} & e^{k\beta}_{33} & 0 & 0 & 0
\end{bmatrix}
\]

where \( e^{k\beta}_{31} = e^{k\beta}_{32} \neq e^{k\beta}_{33} ; e^{k\beta}_{15} = e^{k\beta}_{24} = \left( e^{k\beta}_{33} - e^{k\beta}_{31} \right) / 2 \).

The matrix of the dielectric permittivity tensor \( \varepsilon_{mn} \) has diagonal structure and

\[
\begin{bmatrix}
  \varepsilon_{11} & 0 & 0 \\
  0 & \varepsilon_{22} & 0 \\
  0 & 0 & \varepsilon_{33}
\end{bmatrix}
\]

where \( \varepsilon_{11} = \varepsilon_{22} \neq \varepsilon_{33} \).

Let us assume that the thickness of the electrodes is negligible in comparison with the disk thickness.

On the ring electrode 1 (its width is equal to \( 2d_1 \)) the electrical potential difference \( U_0 e^{i\omega t} \) from a source of electrical signals with the output impedance \( Z_i \) is applied. Obviously, on the electrode 1 we will have another value of the electrical potential \( U_0 e^{i\omega t} \), where \( |U_0| < U_1 \), that can be written as follows

\[
U_0 = \frac{U_1 Z_1}{Z_i + Z_1}.
\]

Electrical impedance \( Z_1 \) can be determined from Ohm’s law for electrical circuit section

\[
Z_1 = \frac{U_0}{I_1}.
\]

If on the surface of the electrode 1 we have harmonically time varying electric charge

\[
q(t) = Q_1 e^{i\omega t},
\]

the electric current amplitude value is determined as follows [20]

\[
I_1 = -j\omega Q_1.
\]

The amplitude value of the electric charge \( Q_1 \) is determined by the axial component \( D_z(\rho, \alpha) \) of the electric induction vector

\[
Q_1 = 2\pi \int_{R_1-d_1}^{R_1+d_1} \rho D_z(\rho, \alpha) d\rho.
\]

Electrical condition of any material object is determined by Maxwell’s equations

\[
\text{rot} \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t},
\]

\[
\text{rot} \vec{E} = -\frac{\partial \vec{B}}{\partial t},
\]

where \( \vec{J} = r \vec{E} \) is a surface density of the conduction current; \( r \) is a specific electric conductivity of the material. Since the piezoelectric ceramic is a fairly good isolator it can be considered that \( r \equiv 0 \). In this case, Maxwell’s equation (8) for harmonically varying fields takes the following form

\[
\text{rot} \vec{H} = j\omega \vec{D}.
\]

Calculating the divergence of the left and right side of (10), we can come to the following conclusion.
\[ \text{div} \mathbf{D} = 0. \quad (11) \]

Equation (11) has the meaning of the condition of absence of free carriers of electricity in a volume of the ideal dielectric.

In [21] it is shown that at a frequency range up to 10 MHz, the magnetic component of the electromagnetic field in a deformable piezoelectric ceramics by several orders less than electrical component. It gives the basis for (9)

\[ \text{rot} \mathbf{E} \cong 0. \quad (12) \]

Equation (12) suggests that the electric field in a volume of the deformed piezoceramics is irrotational, i.e. potential and it can be described by a scalar electric potential, and

\[ \mathbf{E} = - \nabla \Phi. \quad (13) \]

With the definition (13), known [20, 21] expression for calculating of the \( m \)-th electric induction vector component in a volume of a deformable piezoelectric can be written as follows

\[ D_m = e_{\beta \lambda} \varepsilon_{k j} - \chi_{n n}^k \nabla \Phi_n, \quad (14) \]

where \( e_{\beta \lambda} \Leftrightarrow e_{\alpha 0} \) (\( \beta \) is a Voigt index, by which it is changed a couple of symmetrical tensor indices \( k, j \)) is an element of the matrix of piezoelectric constants; \( \varepsilon_{k j} \) is a component of infinitesimal deformations tensor; \( \chi_{n n}^k \) is a component of the dielectric permittivity tensor; \( \nabla \Phi_n \) is the \( n \)-th component of scalar potential gradient vector. When writing the equation (14) in a cylindrical coordinate system we should consider the following correspondence between the symbols \( (\rho, \varphi, z) \) of the coordinate axes of the cylindrical coordinate system and the numbers \( k = 1, 2, 3 \) of the coordinate axes \( x_k \) of the Cartesian coordinate system: \( 1 \Leftrightarrow \rho; \ 2 \Leftrightarrow \varphi; \ 3 \Leftrightarrow z \).

From the general expression (14) the following follows

\[ D_\rho = 2 \varepsilon_{1 3} \varepsilon_{p z} - \chi_{1 1}^z \frac{\partial \Phi^{(1)}(\rho, \varphi, z)}{\partial \rho}, \quad (15) \]

\[ D_z = e_{3 1} \varepsilon_{p p} + e_{3 2} \varepsilon_{q q} + e_{3 3} \varepsilon_{z z} - \chi_{3 3}^z \frac{\partial \Phi^{(1)}(\rho, \varphi, z)}{\partial z} = e_{3 1} \left( \varepsilon_{p p} + \varepsilon_{q q} \right) + e_{3 3} \varepsilon_{z z} - \chi_{3 3}^z \frac{\partial \Phi^{(1)}(\rho, \varphi, z)}{\partial z}, \quad (16) \]

where \( D_\rho = 0 \) because of the axial symmetry of the problem under consideration; \( \varepsilon_{p z} = \frac{\partial u_p / \partial z + \partial u_z / \partial \rho}{2} \) is a shear deformation. In (16) piezoelectric moduli of the same value \( e_{31} \) and \( e_{32} \) (see comment to the matrix (2)) are written, as is usual in solid mechanics, by the same symbol \( e_{31} \). Components \( e_{p p} = \partial u_p / \partial \rho, \ e_{q q} = u_p / \rho \) and \( e_{z z} = \partial u_z / \partial z \) determine compression and expansion deformations along the coordinate lines of a cylindrical coordinate system. \( \Phi^{(1)}(\rho, \varphi, z) \) is an electrical potential in the ring area \( \{ R_1 - d_1 \leq \rho \leq R_1 + d_1; 0 \leq \varphi \leq 2\pi; 0 \leq z \leq \alpha \} \) under the electrode 1.

Expressions (15) and (16) substituting into condition (11) gives a second order differential equation in partial derivatives relative to the required scalar potential \( \Phi^{(1)}(\rho, \varphi, z) \) of the electric field in a deformable piezoelectric.

In the particular case of a sufficiently thin disk when \( \alpha / R < 1 \), it can be argued that in the frequency range in which the length of the elastic wave is larger than the thickness of the piezoelectric disk, electrical and elastic fields in its volume is almost independent of the axial coordinate values \( z \), i.e., practically do not change their values according to thickness of the disk.

If the disk is gently fixed along the surface \( \{ \rho = R; 0 \leq \varphi \leq 2\pi; 0 \leq z \leq \alpha \} \), the shear deformation becomes zero on this surface and on surfaces \( z = 0 \) and \( z = \alpha \). In addition, on the surface covered by the electrode \( z = 0 \) the radial component \( D_\rho = 0 \). The radial component \( D_\rho = 0 \) on the side surface of the piezoceramic disk [21], on the surface of ring electrode 1 and on the disc symmetry axis, i.e. on the axis \( Oz \). The combination of these facts suggests that in thin piezoceramic disk, in a first approximation, it can be considered that \( D_\rho = 0 \forall \{ \rho, \varphi, z \} \in V \), where \( V \) is a volume of the disk. In this case, the vector of electric induction is completely determined by only one non-zero axial component \( D_z \), and the condition (11) takes the form

\[ \frac{\partial D_z^{(1)}(\rho, \varphi, z)}{\partial z} = 0, \quad (17) \]

where \( D_z^{(1)} \) further underlines the fact that we are talking about electric induction vector at the ring area \( \{ R_1 - d_1 \leq \rho \leq R_1 + d_1; 0 \leq \varphi \leq 2\pi; 0 \leq z \leq \alpha \} \) under the electrode 1.

From condition (17) it implies that the axial component \( D_z^{(1)} \) is a function of the radial coordinate \( \rho \) and is independent of the axial coordinate values \( z \).
which is in full agreement with the above mentioned adopted suggestion about a weak dependence of the physical characteristics of the fields on the axial coordinate values in the frequency range in which the following inequality holds \( \lambda \gg \alpha \) (\( \lambda \) is an elastic wave length.

Because of 
\[ e_{pp} + e_{pp} = \frac{\partial u_p}{\partial \rho} + u_p/\rho = [\partial (\rho u_p)/\partial \rho]/\rho, \]
definition (16) can be written as follows
\[ D_{zz}^{(1)}(\rho) = \frac{e_{31}}{\rho} \frac{\partial}{\partial \rho} \left[ \rho u_p^{(i)}(\rho) \right] + e_{33} \frac{\partial u_p^{(i)}}{\partial z} - \chi_{33} \frac{\partial \Phi^{(i)}}{\partial z}, \tag{18} \]
where \( u_p^{(i)}(\rho, z) \) and \( u_z^{(1)}(\rho, z) \) are amplitude values of the components of the material particles displacement vector in the ring area
\[ \{R_1 - d_1 \leq \rho \leq R_1 + d_1; 0 \leq \varphi \leq 2\pi; 0 \leq z \leq \alpha \}. \]

Integrating with respect to \( z \) the left and right side of (18), and taking into account the condition (17), we obtain the following result
\[ \alpha D_{zz}^{(1)}(\rho) = \frac{e_{31}}{\rho} \frac{\partial}{\partial \rho} \left[ \rho u_p^{(i)}(\rho, z)dz \right] + \]
\[ + e_{33} \left[ u_p^{(i)}(\rho, \alpha) - u_p^{(i)}(\rho, 0) \right] - \chi_{33} \left[ \Phi^{(i)}(\alpha) - \Phi^{(i)}(0) \right], \tag{19} \]
Let
\[ u_p^{(i)}(\rho) = \frac{\alpha}{\alpha_0} \int u_p^{(i)}(\rho, z)dz, \tag{20} \]
and \( u_p^{(i)}(\rho) \) is an averaged over the thickness of the disk radial component of the material particles displacement vector in the ring area under the electrode 1. Since \( \Phi^{(i)}(\alpha) - \Phi^{(i)}(0) = U_0 \), then (19) takes the form
\[ \alpha D_{zz}^{(1)}(\rho) = \frac{e_{31}}{\rho} \frac{\partial}{\partial \rho} \left[ \rho u_p^{(i)}(\rho) \right] + \]
\[ + \frac{e_{33}}{\alpha} \left[ u_p^{(i)}(\rho, \alpha) - u_p^{(i)}(\rho, 0) \right] - \chi_{33} \frac{U_0}{\alpha}. \tag{21} \]

Substituting (21) into definition (7) of the amplitude value of electric charge, we can obtain
\[ Q_i = 2\pi \left[ e_{31} \left[ \rho u_p^{(i)}(\rho) \right] \right]_{R_1 + d_1}^{R_1 - d_1} + \]
\[ + \frac{e_{33}}{\alpha} \int_{R_1 - d_1}^{R_1 + d_1} \rho \left[ u_p^{(i)}(\rho, \alpha) - u_p^{(i)}(\rho, 0) \right] d\rho - \]
\[ - \frac{\chi_{33}}{2\alpha} \left( R_1 + d_1 \right)^2 - \left( R_1 - d_1 \right)^2 U_0 \right]. \tag{22} \]

We set
\[ u_p^{(i)}(z) = \frac{1}{2d_1 R_1} \int_{R_1 - d_1}^{R_1 + d_1} \rho u_p^{(i)}(\rho, z) d\rho, \tag{23} \]
where \( u_p^{(i)}(z) \) is an averaged over the area of the ring \( \{R_1 - d_1 \leq \rho \leq R_1 + d_1; 0 \leq \varphi \leq 2\pi \} \) axial component of the material particles \( \bar{u}(\rho, z) \) displacement vector in the ring area under the electrode 1. With the definition (23) relation (22) can be written as follows
\[ Q_i = 2\pi e_{31} \left[ \left( R_1 + d_1 \right) u_p^{(i)}(\rho, z) d\rho \right]_{R_1 - d_1}^{R_1 + d_1} - \]
\[ - \left( R_1 - d_1 \right) u_p^{(i)}(\rho, z) d\rho \right] + \]
\[ + 4\pi d_1 R_1 e_{33} \left[ u_p^{(i)}(\alpha) - u_p^{(i)}(0) \right] - C_{\alpha}^e U_0. \tag{24} \]

where \( C_{\alpha}^e = 4\pi d_1 R_1 \chi_{33}/\alpha \) is a static electric capacity of the piezoceramic volume under the ring electrode No. 1.

Since by definition the piezoelectric transformer is a linear physical system, the averaged components of the material particles displacement vector can always be represented as follows
\[ u_p^{(i)}(\rho) = U_0 F_p^{(i)}(\rho), \quad u_z^{(1)}(\rho) = U_0 F_z^{(1)}(\rho), \tag{25} \]
where functions \( F_p^{(i)}(\rho) \) and \( F_z^{(1)}(\rho) \) differ from the averaged components \( u_p^{(i)}(\rho) \) and \( u_z^{(1)}(\rho) \) of the material particles displacement vector only by a constant factor \( U_0 \), and have the meaning of displacements sensitivity in the ring area \( \{R_1 - d_1 \leq \rho \leq R_1 + d_1; 0 \leq \varphi \leq 2\pi; 0 \leq z \leq \alpha \} \) to the amplitude values of electrical potential difference on the ring electrode 1. The dimension of \( F_p^{(i)}(\rho) \) and \( F_z^{(1)}(\rho) \) is \( \text{m/V} \). Functions \( F_p^{(i)}(\rho) \) and \( F_z^{(1)}(\rho) \) numerically equal to the material particles averaged displacements of the ring area under the electrode 1 when the electric potential difference with the amplitude value of \( U_0 = 1 \text{ V} \) is applied to this electrode.

Following suggestions (25), the expression (24) for the electric charge \( Q_i \) calculation can be written as follows
\[ Q_i = U_0 C_{\alpha}^e F_z^{(i)}(\varphi, \Pi_i), \tag{26} \]
where dimensionless function \( F_z^{(i)}(\varphi, \Pi_i) \) is defined as follows
\[ F_1(\omega, \Pi_1) = \frac{e_{31} \alpha}{2 \chi_{33}^* d_1} \left( 1 + \frac{d_1}{R_1} \right) F_1^{(1)} (R_1 + d_1) - \left( 1 - \frac{d_1}{R_1} \right) F_1^{(1)} (R_1 - d_1) + \frac{e_{33}}{\chi_{33}^*} \left[ F_1^{(1)} (\alpha) - F_1^{(1)} (0) \right] - 1. \] (27)

Substituting (26) into the definition (6) of the electric current amplitude, and the obtained result into Ohm’s law (5) for the circuit section, we can get the estimated ratio for the electrical impedance \( Z_1 \):

\[ Z_1 = \frac{1}{\text{io} C_1^* F_1(\omega, \Pi_1)}. \] (28)

If the dielectric under the ring electrode 1 does not have piezoelectric properties, i.e. \( e_{31} = e_{33} = 0 \), the function \( F_1(\omega, \Pi_1) = -1 \) and the expression (28) becomes as well-known formula for capacitor reactive resistance calculation with capacitance \( C_1^* \), i.e.

\[ Z_1 = \frac{1}{\text{io} C_1^*}. \]

Substituting (28) into the formula (4), we obtain

\[ U_0 = \frac{U_I}{1 - \text{io} C_1^* F_1(\omega, \Pi_1)} Z_i. \] (29)

It should be emphasized that the potential difference \( U_0 \) is determined by components averaged values of the material particles displacement vector of the ring area \( \{R_1 - d_1 \leq \rho \leq R_1 + d_1; 0 \leq \varphi \leq 2\pi; 0 \leq z \leq \alpha \} \). This fact is of fundamental importance, since there is the possibility of equations joint solutions of a deformable piezoelectric motion.

In the case when a strong inequality \( \alpha/R << 1 \) takes place, i.e. when the disk can be considered as infinitely thin, the situation is considerably simplified, since the deformation \( \varepsilon_{zz} \) becomes linearly dependent on the sum of deformations \( \varepsilon_{pp} \) and \( \varepsilon_{qq} \).

From the generalized Hooke’s law [20] for the elastic media with piezoelectric properties

\[ \sigma_{ij} = e_{ijkl} \varepsilon_{kl} + e_{ijl} \frac{\partial \Phi}{\partial x_l}, \]

where \( \sigma_{ij} \) is a component of the resulting mechanical stresses tensor, follows that in a polarized across the thickness piezoceramic disk normal stresses \( \sigma_{pp}, \sigma_{qq} \) and \( \sigma_{zz} \) correspond to compression and expansion deformations \( \varepsilon_{pp}, \varepsilon_{qq} \) and \( \varepsilon_{zz} \) and can be defined by the following expressions:

\[ \sigma_{pp} = c_{11}^E \varepsilon_{pp} + c_{12}^E \varepsilon_{qq} + c_{31}^E \varepsilon_{zz} + \varepsilon_{31} \frac{\partial \Phi}{\partial z}, \] (30)
\[ \sigma_{qq} = c_{12}^E \varepsilon_{pp} + c_{11}^E \varepsilon_{qq} + c_{12}^E \varepsilon_{zz} + \varepsilon_{31} \frac{\partial \Phi}{\partial z}, \] (31)
\[ \sigma_{zz} = c_{13}^E (\varepsilon_{pp} + \varepsilon_{qq}) + c_{33}^E \varepsilon_{zz} + \varepsilon_{33} \frac{\partial \Phi}{\partial z}. \] (32)

In expressions (30)–(32) material constants of the same value (the elements of matrices (1) and (2)) are written by the same symbols.

On the bottom (\( z = 0 \)) and top (\( z = \alpha \)) surfaces of the piezoceramic disk free from mechanical contacts with other material objects in accordance with Newton’s third law the following conditions should take place:

\[ \varepsilon_{zp} \vert_{z=0;\alpha} = \sigma_{zz} \vert_{z=0;\alpha} = 0. \] (33)

Since the disk is very thin, it can be argued that the quantitative characteristics of its stress-strain state does not depend on the axial coordinate values \( z \), i.e. \( \partial \sigma_{ij}/\partial z \approx 0 \). It follows that the condition (33) must be satisfied at any point of the volume \( V \) of a thin piezoceramic disk. Substituting into the left side of (32) a zero, we obtain the following definition for the compression and expansion deformations in the axial direction:

\[ \varepsilon_{zz} = - \frac{c_{12}^E}{c_{33}^E} \left( \varepsilon_{pp} + \varepsilon_{qq} \right) - \frac{c_{33}^E}{c_{33}^E} \frac{\partial \Phi}{\partial z}. \] (34)

Substituting expression (34) into (30), (31) and (16), it produces the following results:

\[ \sigma_{pp} = c_{11}^E \varepsilon_{pp} + c_{12}^E \varepsilon_{qq} + \varepsilon_{31} \frac{\partial \Phi}{\partial z}, \] (35)
\[ \sigma_{qq} = c_{12}^E \varepsilon_{pp} + c_{11}^E \varepsilon_{qq} + \varepsilon_{31} \frac{\partial \Phi}{\partial z}, \] (36)
\[ D_z = c_{31}^* (\varepsilon_{pp} + \varepsilon_{qq}) - \varepsilon_{33} \frac{\partial \Phi}{\partial z}. \] (37)

where \( c_{11} = c_{11}^E - \left( \frac{c_{12}^E}{c_{33}^E} \right)^2 \); \( c_{12} = c_{12}^E \left( 1 - \frac{c_{12}^E}{c_{33}^E} \right) \); \( e_{31} = \varepsilon_{31} - e_{33} c_{12}/c_{33}^E \) are material constants for planar stress-strain state of the polarized across the thickness piezoceramic element; \( \chi_{33}^* = \chi_{33}^E + \frac{e_{33}^2}{c_{33}^E} \) is a dielectric permittivity of the polarization across the thickness piezoceramic disk for constancy mode (equality...
to zero) of the normal mechanical stresses $\sigma_{zz}$. Equations (35) and (36) in combination with $\sigma_{\varphi z} = \sigma_{z\varphi} = 0 \forall (\rho, z) \in V$ suggest that $u_z \equiv 0$ in the entire oscillating disk.

The expression (29) takes the form

$$U_0 = \frac{U_i}{1 - i\omega C_0' F_1^{(0)} (\omega, \Pi_1) Z_i},$$

where $C_0' = 4\pi d_1 R_1 X_{33}^\prime / \alpha$ is a static electrical capacitance of the ring area of infinitely thin disk under the electrode 1;

$$F_1^{(0)} (\omega, \Pi_1) = \frac{\varepsilon_0' \alpha}{2 X_{33}^\prime d_1} \left[ 1 + \frac{d_1}{R_1} F_\varphi^{(1)} (R_1 + d_1) - \left( 1 - \frac{d_1}{R_1} \right) F_\varphi^{(1)} (R_1 - d_1) \right] - 1.$$  (39)

Now let us consider the processes that occur in an area of the ring electrode 2, i.e. output electrode of the piezoelectric transformer.

Obviously, in the ring area 2 $\{ R_2 - d_2 \leq \rho \leq R_2 + d_2; 0 \leq \varphi \leq 2\pi; 0 \leq z \leq \alpha \}$ the amplitude values $u_\varphi^{(2)} (\rho, z)$ and $u_z^{(2)} (\rho, z)$ of the harmonically time varying components of the material particles displacement vector of the oscillating piezoelectric disk can be represented as follows:

$$u_\varphi^{(2)} (\rho, z) = U_0 F_\varphi^{(2)} (\rho, z),$$
$$u_z^{(2)} (\rho, z) = U_0 F_z^{(2)} (\rho, z),$$

where $U_0$ is an electric potential difference on the exciting ring electrode 1 (Fig. 1); $F_\varphi^{(2)} (\rho, z)$ and $F_z^{(2)} (\rho, z)$ are displacements sensitivities in the ring area 2.

The amplitude value $U_2$ of the voltage drop on electrical load $Z_n$, i.e. on the input impedance of the electronic circuit which is directly connected to the ring electrode 2, is defined as follows

$$U_2 = Z_n I_2,$$  (41)

where $I_2 = -i\omega Q_2$ is an amplitude of the electric current in the conductor, which connects the electrode 2 and the electrical load $Z_n$; $Q_2$ is an amplitude value of the electric charge on the ring electrode 2.

Acting in the same manner as in the determination of the electrical impedance $Z_1$, we can obtain the following definition of the charge $Q_2$:

$$Q_2 = C_2' U_0 F_2 (\omega, \Pi_2) - C_2' U_2,$$  (42)

where $C_2' = 4\pi d_2 R_2 X_{33}^\prime / \alpha$ is a static electrical capacitance of the ring area 2;

$$F_2 (\omega, \Pi_2) = \frac{\varepsilon_0' \alpha}{2 X_{33}^\prime d_2} \left[ 1 + \frac{d_2}{R_2} F_\varphi^{(2)} (R_2 + d_2) - \left( 1 - \frac{d_2}{R_2} \right) F_\varphi^{(2)} (R_2 - d_2) + \varepsilon_3 \chi_{33} \left[ F_z^{(2)} (\alpha) - F_z^{(2)} (0) \right] \right].$$

$F_\varphi^{(2)} (\rho)$ and $F_z^{(2)} (z)$ are averaged sensitivities.

Substituting (42) into current definition $I_2$, and obtained result into (41), we can come to the conclusion that

$$U_2 = f_n (\omega) U_0 F_2 (\omega, \Pi_2),$$  (43)

where $f_n (\omega) = -i\omega C_2' Z_n / \left[ 1 - i\omega C_2' Z_n \right]$ is a switching function on load characteristic of the output ring electrode of the piezoelectric transformer.

In the short-circuit mode ($Z_n = 0$) function $f_n (\omega) = 0$ and $U_2 = 0$. This fact is very clear and does not require any mathematical calculations to prove its validity. In idle mode, when $Z_n \rightarrow \infty$, switching on function $f_n (\omega)$ if $\omega = 0$ is equal to zero, and at an arbitrarily small $\omega > 0$ $f_n (\omega) = 1$, i.e. in this mode switching on function is a function of Heaviside. It follows that the piezoelectric receiver of elastic vibrations is not capable to register the static pressures and deformations. This statement is not so obvious to practitioners actually cancels a large group of devices of piezoelectronics, which are presented in [22].

The rate of change of the switching on function $f_n (\omega)$ is determined by the time constant $\tau_n = C_2' Z_n$ of the circuit that connects the receiver electrode to an electrical load. The function module values $f_n (\omega)$, depending on the value of the dimensionless quantity $\Omega_n = \omega \tau_n$ are shown in Fig. 2.
After substituting (29) into (43), we can write the following definition of the transformation $K(\omega, \Pi)$ of the piezoelectric transformer

$$K(\omega, \Pi) = \frac{U_2}{U_1} = \frac{f_n(\omega) F_2(\omega, \Pi_2)}{1 - \text{io}C_1^{\prime} F_1(\omega, \Pi_1) Z_i}. \quad (44)$$

In the case of very thin piezoceramic disk, when a strong inequality $\alpha/R << 1$ takes place an expression (44) can be written as follows

$$K^{(0)}(\omega, \Pi) = \frac{U_2}{U_1} = \frac{f_n(\omega) F_2^{(0)}(\omega, \Pi_2)}{1 - \text{io}C_1^{\prime} F_1^{(0)}(\omega, \Pi_1) Z_i}. \quad (45)$$

where $f_n(\omega) = -\text{io}C_2^{\prime\prime} Z_n/(1 - \text{io}C_2^{\prime\prime} Z_n)$; $C_2^{\prime\prime} = 4nd_2 R_2 \gamma_{33}/\alpha$;

$$F_2^{(0)}(\omega, \Pi_2) = \frac{\varepsilon^1_0 \alpha}{2 \gamma_{33} d_2^2} \left[1 + \frac{d_2}{R_2}\right] F_p(\rho, R_2 - d_2) - \left[1 - \frac{d_2}{R_2}\right] F_p(\rho, R_2 + d_2); \quad F_1^{(0)}(\omega, \Pi_1)$$

is defined by (39).

Expressions (44) and (45), which have a sense of mathematical models of piezoelectric transformers operating on axially symmetric radial oscillations of piezoceramic disks, are built with a minimal number of simplifying assumptions.

To fill the definition (44) or (45) by a specific physical meaning, it is necessary to determine the components of the material particles displacement vector of the oscillating piezoceramic disk. This procedure is the subject of a separate investigation.

4 EXPERIMENTS

Let us consider a disk piezoelectric transformer (Fig. 3), primary electrical circuit of which consists of electric potential difference generator $U_1 e^{\text{ext}}$ (where $U_1$ is an amplitude value of electric potential difference) with output electrical impedance $Z_g$ and ring electrode (position 1 in Fig. 3). The secondary electrical circuit consists of an electrode in the form of a circle (position 2) with connected electronic circuit to it with input electrical impedance $Z_n$, on which an electric potential difference $U_2 e^{\text{ext}}$ is formed. The primary and secondary circuits of piezoelectric transformer do not have a galvanic connection. The energy exchange between the primary and secondary circuits is carried out by means of axisymmetric radial vibrations of the piezoceramics material particles in the volume of thickness polarized disk (position 3 in Fig. 3).

It is obvious that the work of function piezoelectronic element, which is schematically shown in Fig. 3, is fully described by transformation ratio $K(\omega, \Pi) = U_2/U_1$, which is a mathematical model of the device under consideration. Scheme of construction of piezoelectric transformer’s mathematical model is outlined in [23].

The elastic stresses and displacements of material particles of piezoelectric ceramics in the areas under the electrodes, and in the areas where there are no electrodes are determined in [24]. Following the method which is described in [24] we can write that

$$\sigma_{pp}^{(i)}(\rho) = c_{11} \frac{\partial u_{p}^{(i)}(\rho)}{\partial \rho} + c_{12} \frac{u_{p}^{(i)}(\rho)}{\rho} + c_{31}^* \frac{U_2}{\alpha}, \quad (46)$$

$$\sigma_{pp}^{(2)}(\rho) = c_{11} \frac{\partial u_{p}^{(2)}(\rho)}{\partial \rho} + c_{12}^D \frac{u_{p}^{(2)}(\rho)}{\rho}, \quad (47)$$

$$\sigma_{pp}^{(3)}(\rho) = c_{11} \frac{\partial u_{p}^{(3)}(\rho)}{\partial \rho} + c_{12} \frac{u_{p}^{(3)}(\rho)}{\rho} + c_{31}^* \frac{U_0}{\alpha}, \quad (48)$$

$$\sigma_{pp}^{(4)}(\rho) = c_{11} \frac{\partial u_{p}^{(4)}(\rho)}{\partial \rho} + c_{12}^D \frac{u_{p}^{(4)}(\rho)}{\rho}, \quad (49)$$

where $c_{11} = c_{11}^E - (c_{12}^E)^2/c_{33}^E$; $c_{12} = c_{12}^E (1 - c_{12}^E/c_{33}^E)$; $c_{11}^D = c_{11} + (c_{31}^E)^2/c_{13}^E$; $c_{12}^D = c_{12} + (c_{31}^E)^2/c_{13}^E$ are moduli of elasticity for the mode of axially symmetric radial oscillations of the piezoceramic disk material particles in the areas under the electrodes (area No.1, where $\rho \in [0, R_1]$, and area No.3, where $\rho \in [R_2, R_3]$) and in the areas where there are no electrodes (area No.2, where $\rho \in [R_1, R_2]$), and area No.4, where $\rho \in [R_3, R_4]$).

The amplitude values of the radial components of the material particles displacements vectors in the areas No.1, ..., No.4, are defined as follows:

$$u_{p}^{(i)}(\rho) = A_i J_i (\gamma_\rho), \quad (50)$$

$$u_{p}^{(2)}(\rho) = A_2 J_1 (\gamma_1 \rho) + A_3 J_3 (\gamma_3 \rho), \quad (51)$$
\[ u^{(3)}_0(\rho) = A_4 J_1(\gamma \rho) + A_5 N_1(\gamma \rho), \]
\[ u^{(4)}_0(\rho) = A_6 J_1(\gamma_1 \rho) + A_7 N_1(\gamma_1 \rho), \]

where \( A_1, \ldots, A_7 \) are frequency-dependent constants of the radial displacements of material particles in various areas; \( J_1(z), N_1(z) \) \((z = \gamma \rho; z = \gamma_1 \rho)\) are Bessel and Neumann functions \([25]\) of the first order; \( \gamma = \omega / \sqrt{c_{11}/\rho_0} \) and \( \gamma_1 = \omega / \sqrt{c_{11}/\rho_0} \) are wave numbers of the radial oscillations in the areas under the electrodes, and in the areas where there are no electrodes; \( \rho_0 \) is a piezoceramics density.

In the conditional separation boundaries the amplitudes of displacements and stresses should satisfy the conditions of dynamic and kinematic coupling, which can be written as follows:

\[ \sigma^{(1)}_{pp}(R_i) - \sigma^{(2)}_{pp}(R_i) = 0, \quad (54) \]
\[ u^{(1)}_p(R_i) - u^{(2)}_p(R_i) = 0, \quad (55) \]
\[ \sigma^{(2)}_{pp}(R_2) - \sigma^{(3)}_{pp}(R_2) = 0, \quad (56) \]
\[ u^{(2)}_p(R_2) - u^{(3)}_p(R_2) = 0, \quad (57) \]
\[ \sigma^{(3)}_{pp}(R_3) - \sigma^{(4)}_{pp}(R_3) = 0, \quad (58) \]
\[ u^{(3)}_p(R_3) - u^{(4)}_p(R_3) = 0. \quad (59) \]

If boundary \( \rho = R \) of the piezoceramic disk is free from mechanical contacts with other material objects, then on the contour \( \rho = R \) next condition should be satisfied

\[ \sigma^{(4)}_{pp}(R) = 0. \quad (60) \]

Substituting expressions (46)–(53) into conditions (54)–(60), we obtain an inhomogeneous system of linear algebraic equations, which consists of seven equations, that contain seven sought constants \( A_1, \ldots, A_7 \). It is obvious that this system of equations is solved in one way. In general terms, mentioned system of equations can be written as follows:

\[ \sum_{k=1}^{7} m_{jk} A_k = P_j, \quad (j, k = 1, 2, \ldots, 7). \quad (61) \]

The coefficients \( m_{jk} \) and right-hand parts \( P_j \) of equations (61) have the following form:

\[ m_{1j} = J_0(\gamma R_i) - (1 - k_j) J_1(\gamma R_i) / (\gamma R_i); \quad k = c_{12}/c_{11}; \]

\[ m_{12} = \xi J_0(\gamma_1 R_i) - (1 - k_1) J_1(\gamma_1 R_i) / (\gamma_1 R_i); \]

\[ m_{13} = \xi(N_0(\gamma_1 R_i) - (1 - k_1) N_1(\gamma_1 R_i) / (\gamma_1 R_i)); \]

\[ m_{14} = m_{15} = m_{16} = m_{17} = 0; \quad P_1 = \gamma U_0/\Omega; \quad q = e_{13} R / (c_{13} \alpha); \quad \Omega = \gamma R; \quad m_{21} = J_1(\gamma R_i); \]

\[ m_{22} = J_0(\gamma R_i); \quad m_{31} = 0; \quad m_{23} = N_1(\gamma R_i); \]

\[ m_{24} = m_{25} = m_{26} = 0; \quad P_2 = 0; \]

\[ m_{32} = \xi J_0(\gamma R_i) - (1 - k_1) N_1(\gamma R_i) / (\gamma R_i); \]

\[ m_{33} = \xi(N_0(\gamma R_i) - (1 - k_1) N_1(\gamma R_i) / (\gamma R_i)); \]

\[ m_{34} = J_0(\gamma R_i) - (1 - k_1) J_1(\gamma R_i) / (\gamma R_i); \]

\[ m_{35} = N_0(\gamma R_i) - (1 - k_1) N_1(\gamma R_i) / (\gamma R_i); \]

\[ m_{36} = m_{37} = 0; \quad P_3 = \gamma U_0/\Omega; \quad m_{41} = m_{42} = J_1(\gamma R_i); \]

\[ m_{43} = N_1(\gamma R_i); \quad m_{44} = J_1(\gamma R_i); \quad m_{45} = N_1(\gamma R_i); \]

\[ m_{46} = m_{47} = 0; \quad P_4 = 0; \quad m_{51} = m_{52} = m_{53} = 0; \]

\[ m_{54} = J_0(\gamma R_i) - (1 - k_1) J_1(\gamma R_i) / (\gamma R_i); \]

\[ m_{55} = N_0(\gamma R_i) - (1 - k_1) N_1(\gamma R_i) / (\gamma R_i); \]

\[ m_{56} = \xi J_0(\gamma R_i) - (1 - k_1) J_1(\gamma R_i) / (\gamma R_i); \]

\[ m_{57} = \xi(N_0(\gamma R_i) - (1 - k_1) N_1(\gamma R_i) / (\gamma R_i)); \]

\[ P_5 = -\gamma U_0/\Omega; \quad m_{61} = m_{62} = m_{63} = 0; \quad m_{64} = J_1(\gamma R_i); \]

\[ m_{65} = N_1(\gamma R_i); \quad m_{66} = J_1(\gamma R_3); \quad m_{67} = N_1(\gamma R_3); \]

\[ P_6 = 0; \quad m_{71} = m_{72} = m_{73} = m_{74} = m_{75} = 0; \]

\[ m_{76} = J_0(\gamma_1 R_i) - (1 - k_1) J_1(\gamma_1 R_i) / (\gamma_1 R_i); \]

\[ m_{77} = N_0(\gamma_1 R_i) - (1 - k_1) N_1(\gamma_1 R_i) / (\gamma_1 R_i); \quad P_7 = 0. \]

Solutions for constants \( A_1, A_4 \) and \( A_5 \), that define the radial displacements of disk material particles under the electrodes of primary and secondary electrical circuits of piezoelectric transformer are as follows:

\[ A_1 = \frac{q}{\Omega} (U_2 A_{11} + U_0 A_{22}); \quad A_1 = \frac{B_{11}}{D_0}; \quad A_2 = \frac{B_{12}}{D_0}; \quad (62) \]

\[ A_4 = \frac{q}{\Omega} (U_2 A_{41} + U_0 A_{42}); \quad A_4 = \frac{B_{41}}{D_0}; \quad A_4 = \frac{B_{42}}{D_0}; \quad (63) \]

\[ A_5 = \frac{q}{\Omega} (U_2 A_{51} + U_0 A_{52}); \quad A_5 = \frac{B_{51}}{D_0}; \quad A_5 = \frac{B_{52}}{D_0}; \quad (64) \]

where \( D_0 \) is a determinant of the system of equations (61), and \( B_{11}, \ldots, B_{52} \) are determinants of the following matrices:

\[ B_{11} \]

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Substituting definition (62) of the constant $A_i$ into the equation (50), and obtained result into the formula for potential calculating $U_2$ we can come to the conclusion that

$$U_2 = 2f_e (\omega) \frac{e^{\gamma}}{\chi}\Omega_{31}(U_{2A_1} + U_{0A_2})J_1 (\gamma R_i),$$

which implies that

$$U_2 = U_0K_2 (\Omega, \Pi);$$

$$K_2 (\Omega, \Pi) = \frac{2f_e (\omega)K_{2A_2}J_{1}(\Omega \omega R_i)/(\Omega \omega R_i)}{1 - 2f_e (\omega)K_{31}A_{11}J_{1}(\Omega \omega R_i)/(\Omega \omega R_i)};$$

where $K_{2}^{2} = \left(\frac{e^{\gamma}}{\chi}\right)^{2} \left(\gamma_{11}\right)^{2}$ is a squared electromechanical coupling coefficient for the mode of
radial oscillations of thickness polarized piezoceramic disk material particles.

Let us define the amplitude value \( U_0 \) of electric potential difference on the electrode of the piezoelectric transformer’s primary electric circuit.

It is obvious that

\[
U_0 = \frac{U_1 Z_3}{Z_g + Z_3},
\]

(66)

where \( Z_3 \) is an electric impedance of the area No.3 under the ring electrode of the piezoelectric transformer’s primary electric circuit. In accordance with Ohm’s law for the electrical circuit section \( Z_3 = U_0/I_3 \), where \( I_3 \) is an amplitude of the alternating current in the conductor, which connects the generator of electrical potential difference with the ring electrode. As before, we assume that \( I_3 = -i\omega Q_3 \), where \( Q_3 \) is an amplitude value of polarization charge under the ring electrode, which is defined as follows:

\[
Q_3 = 2\pi \int_0^{\infty} \rho \rho_z^{(3)}(\rho) \, d\rho = C_3^2 U_0 \left\{ \frac{2\alpha e_{31}}{R_3 \beta_{33}} \left( 1 - \beta^2 \right) U_0 \left[ u_{\rho}^{(3)}(R_3) - u_{\rho}^{(3)}(R_2) \right] - 1 \right\},
\]

where \( C_3^2 = \pi \beta_{33} (R_3^2 - R_2^2)/\alpha \) is a static electrical capacitance of the ring electrode; \( \beta = R_2/R_3 \) is a geometrical parameter of the ring.

Substituting (63) and (64) for the calculation of constants \( A_4 \) and \( A_5 \) into definition (52), and taking into account the expression (65), we obtain the following formula for the calculation of displacements \( u_{\rho}^{(3)}(\rho) \):

\[
u_{\rho}^{(3)}(\rho) = \frac{U_0 \text{Re} \rho}{\text{C}_{11} \alpha} \left[ \left[ K_2(\Omega,\Pi) A_{41} + A_{42} \right] J_1(\Omega \rho/R) + \left[ K_2(\Omega,\Pi) A_{51} + A_{52} \right] N_1(\Omega \rho/R) \right].
\]

(68)

After calculating the values \( u_{\rho}^{(3)}(R_2) \) and \( u_{\rho}^{(3)}(R_1) \) according to the formula (68) it can be written that \( Q_3 = C_3^2 U_0 K_3(\Omega,\Pi) \), where

\[
K_3(\Omega,\Pi) = \frac{2K_2^2}{1 - \beta^2} \left[ \left[ K_2(\Omega,\Pi) A_{41} + A_{42} \right] J(\Omega) + \left[ K_2(\Omega,\Pi) A_{51} + A_{52} \right] N(\Omega) \right] - 1;
\]

\[
J(\Omega) = \left[ J_1(\Omega R_1/R) - \beta J_1(\beta \Omega R_1/R) \right] / (\Omega R_1/R);
\]

\[
N(\Omega) = \left[ N_1(\Omega R_1/R) - \beta N_1(\beta \Omega R_1/R) \right] / (\Omega R_1/R).
\]

After charge determining \( Q_3 \) the electrical impedance \( Z_3 \) is determined by the expression

\[
Z_3 = -\frac{U_1}{1 - i\omega C_0^2 Z_g K_3(\Omega,\Pi)}.
\]

(69)

Substituting (69) into (65) we can come to the conclusion that

\[
U_2 = \frac{U_1 K_3(\Omega,\Pi)}{1 - i\omega C_0^2 Z_g K_3(\Omega,\Pi)},
\]

from which the formula for the transfer ratio calculation follows

\[
K(\omega,\Pi) = \frac{U_2}{U_1} = \frac{K_3(\Omega,\Pi)}{1 - i\omega C_0^2 Z_g K_3(\Omega,\Pi)}.
\]

(70)

Analytical structure (70) is a mathematical model of piezoelectric ring-dot transformer with ring electrode in the primary circuit.

### 5 RESULTS

Expression (70), which determines the transfer ratio of piezoelectric device, has a structure which is typical for electronic devices with negative feedback. It is clearly seen that the depth of feedback is directly proportional to the value of the signal source output impedance \( Z_g \). If the value of \( Z_g = 0 \) the feedback disappears and transfer ratio is completely determined by a frequency dependent function \( K_3(\Omega,\Pi) \).

Feedback physical content which exists in piezoelectric transformers is practically obvious. Displacements levels of piezoelectric disk material particles increases significantly at a frequency of electromechanical resonance of radial oscillations. This is accompanied by an increase of deformations and as a consequence, by an increase of levels of polarization charges on the electrodes of the primary electrical circuit. Because of this the amplitude of the electric current in the primary circuit increases, which is accompanied by an increase of voltage drop on the resistance \( Z_g \) and, accordingly, by a decrease of potential difference \( U_0 \) (see, Fig. 3).

The transfer ratio modeling of piezoelectric transformer according to (70) have been conducted, the results of which are shown in Fig. 4. As follows from the results shown in Fig. 4, the parameter change \( Z_g \) is accompanied by significant changes in the frequency characteristic of piezoceramic disk transformer.
Fig. 5 illustrates an influence of mechanical $Q$ factor of disk material on a change of transformation ratio in a narrow band near the first electro-mechanical resonance of the radial oscillations of free (not fixed) piezoceramic disk. The numerical values of quality factor are indicated near the corresponding curves.

All calculations were performed for piezoceramic disk with radius $R = 33 \cdot 10^{-3}$ m and thickness $\alpha = 3 \cdot 10^{-3}$ m, made of thickness polarized PZT type piezoceramics with following parameters:

- $\rho_0 = 7400$ kg/m$^3$; $c_{11} = 112$ GPa; $c_{12} = 62$ GPa;
- $c_{33} = 100$ GPa; $e_{33} = 20$ C/m$^2$; $e_{31} = -9$ C/m$^2$;
- $\chi_3 = 1800 \chi_0$; $\chi_0 = 8.85 \cdot 10^{-12}$ F/m is a dielectric constant; $Q = 100$ is a quality factor of piezoceramics;
- $Z_n = 10$ kOhms is an electrical load value; $\Omega = \omega \tau_0$ is a dimensionless quantity, where $\tau_0 = R / \sqrt{\rho_0 c_{11}}$ is a piezoceramic disk time constant. The frequency $f = 15206$ Hz corresponds to the value $\Omega = 1$. The value of the electrical impedance module of the electrical signal source is shown in the figures field.

From the results shown in Fig. 4, 5 it can be concluded that each set of physical and mechanical piezoelectric parameters, each primary and secondary circuit electrodes configuration and fixed electrical load of piezoelectric transformer is corresponded to a fixed value of electrical signal source output impedance $Z_g$, with which the maximum transfer ratio is realized in a specified frequency range.

In Fig. 6 it is shown the calculated (solid line) and the experimentally obtained (dashed line) curves of the frequency dependence of the modulus of piezoceramic ring-dot disk transformer’s transformation coefficient. The calculation is based on the same parameters as in the calculation of the curves $K(\Omega, \Pi)$ shown in Fig. 4. Naturally, the dimensions of the disk transformer in the calculation and experiment are chosen to be the same, i.e., the radius $R = 33 \cdot 10^{-3}$ m, the thickness $\alpha = 3 \cdot 10^{-3}$ m and $R_1/R = 12/25, R_2/R = 15/25$ and $R_3/R = 0.999$:

- a – $Z_g = 5$ Ohms; b – $Z_g = 10$ Ohms;
- c – $Z_g = 20$ Ohms; d – $Z_g = 50$ Ohms

The values of the modulus of transformation coefficient of the piezoceramic disk transformer are plotted along the ordinate axis, and the frequency $f$ (dimensionless value $\Omega$) – on the abscissa axis. The frequency $f = 15206$ Hz corresponds to the value $\Omega = 1$. 
When building the model, it was assumed that the thickness of the electrodes located on the surfaces of the disk is very small in comparison with the thickness of the disk $\alpha$. In other words, the thickness of the electrodes, which, as a rule, does not exceed $15 \, \mu m$, was not taken into account for constructing a mathematical model of piezoelectric transformer based on piezoceramic thin disk ($\alpha/R \ll 1$). It should also be noted that mathematical model (70) was built for ring-dot piezoelectric transformer (see Fig. 3) with surfaces partially covered by electrodes (area 1, $\rho \in [0, R_1]$ and area 3, where $\rho \in [R_2, R_3]$) and in the areas where there are no electrodes (area 2, where $\rho \in [R_1, R_2]$ and area 4, where $\rho \in [R_3, R]$).

As expected, the absolute values of the frequencies of resonances in calculation and experiment differ from each other. So, following the calculation, the frequencies of the first second and third electromechanical resonances are respectively equal to $f_1 = 37193 \, Hz$, $f_2 = 88194 \, Hz$ and $f_3 = 135330 \, Hz$; the frequency ratio $\zeta = f_2/f_1 = 2.371$.

The experimental values of the same quantities are, respectively, $f_1 = 34491 \, Hz$, $f_2 = 83728 \, Hz$, $f_3 = 132325 \, Hz$ and $\zeta = f_2/f_1 = 2.428$. If the experimental data are assumed to be true, the error in determining the frequency ratio is $\Delta \zeta = 2.3\%$. The obtained results are explained very simply. The numerical values of the frequencies of resonances $s$ are determined by the dimensions and physicomechanical parameters of the material of disk element. The ratio of the resonances frequencies of the same disk is determined practically only by its dimensions. For this reason, a very satisfactory match between the theoretically and experimentally determined resonance frequency ratios is observed. The discrepancy between the absolute values of the resonance frequencies is explained by the discrepancy between the physicomechanical parameters of the piezoceramics, which were incorporated into the calculation and which are inherent in the experimentally investigated object. Comparing the curves, we can conclude that the quality factor of the material of the experimentally investigated sample is at least 1.2 times larger than included in the quality factor calculation.

Thus, it can be asserted that the character of the variation of both curves, shown in Fig. 6, in a fairly wide frequency range coincides with accuracy to details. This means that the qualitative content of the expression (70) is adequate to the processes that occur in real object. In other words, expression (70) is a mathematical model of piezoelectric ring-dot transformer with ring electrode in primary electrical circuit and sufficiently adequate to the real object and the processes occurring in it. The latter allows us to assume that the mathematical description of the stress-strain state of the disk transformer also corresponds quite well to the real state of things.

**CONCLUSIONS**

Physical processes in piezoelectric transformers, which operate using axially symmetric radial oscillations of the piezoceramic disk, are considered. The scheme of mathematical models constructing of the ring-dot piezoelectric transformer that is sufficiently adequate to real objects and occurring physical processes is proposed.

Main results of this work can be formulated as follows:
- mathematical model of piezoelectric transformer with ring electrode in the primary electrical circuit is constructed;
- high sensitivity of frequency characteristic of piezoelectric transformer to the values of the output impedance of the electrical signal source in the primary electrical circuit is demonstrated.

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Показники можуть технічних систем осесиметричних реальную становити реальному процесі, фізичним технологіям.

Висновки. У результаті дослідження математичної моделі реального пристрою можна визначити нові відомості, фізично механічних та електричних параметрів реального об’єкта, які забезпечують реалізацію технічних показників функціонального елемента п’єзоелектроники, обумовлених в технічному завданні. Вартість збережених ресурсів становить комерційну ціну математичної моделі. Перспективи подальших досліджень можуть полягати в побудові математичної моделі п’єзоелектричного трансформатора з секторними електродами.

**КЛЮЧОВІ СЛОВА:** п’єзоелектричний трансформатор, висесиметричні коливання, фізичні процеси, математична модель.

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**ПРИНЦИПИ І МЕТОДИ РАСЧЕТА ПЕРЕДАТОЧНЫХ ХАРАКТЕРИСТИК ДИСКОВЫХ ПЬЕЗОЭЛЕКТРИЧЕСКИХ ТРАНСФОРМАТОРОВ**

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**АННОТАЦИЯ**

Актуальность. Благодаря своим уникальным свойствам пьезокерамика находит применение в различных областях техники и технологии. Дисковые пьезоэлектрические устройства широко используются в элементах информационных систем. Исследования показали, что пьезоэлектрические трансформаторы могут конкурировать с традиционными электромагнитными трансформаторами как по эффективности, так и по плотности мощности. Конечной целью математического моделирования физического состояния колеблющихся пьезокерамических элементов является качественное и количественное описание характеристик и параметров существующих в них электрических и упругих полей.

Цель работы – предложить принципы построения математических моделей, которые в достаточной мере адекватны реальным устройствам и происходящим в них физическим процессам.

Метод. Математические модели пьезоэлектрических трансформаторов, работающих с использованием осесимметричных радиальных колебаний пьезокерамических дисков, построены с минимальным числом упрощающих реальную ситуацию предположений. Это позволяет утверждать, что предложенная схема построения дает математические модели, которые в достаточной мере адекватны реальным объектам и физическим процессам, которые в них существуют.

Результаты. Основные результаты настоящей работы можно сформулировать следующим образом: построена математическая модель пьезоэлектрического трансформатора с кольцевым электродом в первичной электрической цепи; показана высокая чувствительность частотной характеристики пьезоэлектрического трансформатора к изменениям значений выходного сопротивления источника электрического сигнала в первичной электрической цепи; выведено выражение для определения параметров реального объекта, который обеспечивает реализацию технических показателей функционального элемента пьезоэлектроники, оговоренных в техническом задании. Стоимость сохраненных ресурсов составляет коммерческую цену математической модели. Перспективы дальнейших исследований могут заключаться в построении математической модели пьезоэлектрического трансформатора с секторными электродами.

**КЛЮЧЕВЫЕ СЛОВА:** пьезоэлектрический трансформатор, осесимметричные колебания, физические процессы, математическая модель.

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